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GEODETIC HUB NUMBER OF GRAPHS

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Abstract: A subset S of vertex set $V(G)$ of a graph G of order $n \geq 2$ is a geodetic hub set of G if S is both a geodetic set and a hub set of G. The minimum cardinality of a geodetic hub set of G is called the geodetic hub number of G, denoted by $h_{geo}(G)$. In this paper, we initiate the study of geodetic hub number of a graph. The geodetic hub number of several classes of graphs are determined and its value for some graph operations are studied.

Keywords and Phrases: Hub set, Hub number, Geodetic set, Geodetic number.

2020 Mathematics Subject Classification: 05C07, 05C69, 05C90.

1. Introduction

By a graph $G = (V, E)$ we mean a simple, connected undirected graph without loops and multiple edges. $|V(G)| = n$ and $|E(G)| = m$ denote number of vertices and edges of G, respectively. The distance $d(u, v)$ between two vertices u and v in G is the length of the shortest path joining them. The shortest $u - v$ path is called a geodesic. The diameter of a graph is the length of any longest geodesic, denoted by $d(G)$. For basic graph terminology and definitions not given here we refer Harary F., [3].

A vertex v is extreme vertex of G if the subgraph induced by its neighbors is a complete graph. This was introduced in Everett M. G., and Seidman S. B., [2]. The geodetic closure of a vertex set $S \subset V(G)$ is the set of all vertices $u \in V(G)$ which lie in some geodesic in G joining two vertices x and y of S. The geodetic number of G denoted by $g(G)$, is the minimum number of vertices in a set S whose closure is all of V. This was introduced by Harary *et al* [4] and it can find applications in location theory and convexity theory, they proved that determination of $g(G)$ is an NP-hard problem and its decision problem is NP-complete.

The concept of hub number was introduced by Walsh M., [8], a subset $H \subseteq V(G)$ is a hub set of G if for any two vertices outside H there exists a path with all internal vertices in H(This includes the degenerate cases where the path consists of single edge uv or a single vertex u if $u = v$; such an $H-$path is trivial). A hub set H of G is a minimal hub set of G if $H \setminus \{v\}$ is not a hub set of G, for any $v \in H$. The minimum cardinality of a minimal hub set is hub number of G and is denoted by $h(G)$. Different variations of hub parameters have been studied in Khalaf, S. I., and Mathad, V., [5, 6] and also in Mathad, V., Anand and Puneeth, S., [7].

Theorem 1.1. [1] *Every geodetic set of a graph contains all its extreme vertices.*

Theorem 1.2. [1] *The geodetic number of a tree T is the number of end vertices in T.*

2. Main Results

Definition 2.1. *A subset S of vertex set $V(G)$ is a geodetic hub set of G if S is both a geodetic set and a hub set of G. The minimum cardinality of a geodetic hub set of G is called the geodetic hub number of G, denoted by $h_{geo}(G)$.*

It is obvious that $h(G) \leq h_{geo}(G)$. We first determine the geodetic hub number of some standard graphs.

Observation 1. *1. For any path P_n,* $h_{geo}(P_n) = \begin{cases} 2, \text{ if } n = 2, 3; \\ n-2, \text{ if } n \geq 4. \end{cases}$

2. For any cycle C_n, $h_{geo}(C_n) = \begin{cases} 3, \text{ if } n = 3, 5; \\ 2, \text{ if } n = 4; \\ n-3, \text{ if } n \geq 6 \end{cases}$

3. For any star $K_{1,n}$, $n \geq 2$, $h_{geo}(K_{1,n}) = n$.

4. For complete graph K_n, $h_{geo}(K_n) = n$.

5. For complete bipartite graph $K_{m,n}$, $2 \leq m \leq n$, $h_{geo}(K_{m,n}) = \begin{cases} 2, \text{ if } m \text{ or } n = 2; \\ 3, \text{ if } m \neq 2 \text{ and } n = 3; \\ 4, \text{ if } m, n \geq 4. \end{cases}$

6. For any Wheel graph W_n, $h_{geo}(W_n) = \begin{cases} 4, \text{ if } n = 4; \\ \left\lfloor \dfrac{n}{2} \right\rfloor, \text{ if } n \geq 5. \end{cases}$

Theorem 2.2. *For a connected graph G, $2 \leq max\{h(G), g(G)\} \leq h_{geo}(G) \leq n$.*
Proof. Any geodetic set of G requires at least two vertices. $2 \leq max\{h(G), g(G)\}$. From the definition of geodetic hub number of G, $max\{h(G), g(G)\} \leq h_{geo}(G)$. $V(G)$ is also a geodetic hub set of G, so, $h_{geo}(G) \leq n$. Thus $2 \leq max\{h(G), g(G)\} \leq h_{geo}(G) \leq n$.

Remark 2.3. *The bound in the above theorem is sharp. For lower bound consider C_4, $h(C_4) = 1$, $g(C_4) = 2$ and $h_{geo}(C_4) = 2$. For upper bound consider $K_n (n \geq 2)$, $h_{geo}(K_n) = n$.*

Theorem 2.4. *Every extreme vertex of a connected graph G belongs to every geodetic hub set of G.*
Proof. By the definition of geodetic hub set, every geodetic hub set of G is a geodetic set of G. Hence result holds by the Theorem 1.1.

Theorem 2.5. *For a connected graph G of order $n \geq 2$, $h_{geo}(G) = n$ if and only if $G \cong K_n$.*
Proof. Let $G \cong K_n$. Then every vertex of G is an extreme vertex. Thus $h_{geo}(G) = n$. Conversely, let $h_{geo}(G) = n$, $n \geq 2$. For $n = 2$, $G = K_2$. Let $n \geq 3$. Contrarily suppose there exist nonadjacent vertices u and v in G. Let w be a vertex adjacent to u lying on a $u - v$ geodesic path. Then $V(G) \setminus \{w\}$ is a geodetic hub set of G, a contradiction to $h_{geo}(G) = n$. Thus $G \cong K_n$.

Theorem 2.6. *If G is a connected graph of order $n \geq 2$ with $h_{geo}(G) = 2$ then $d(G) \leq 3$.*
Proof. Let $h_{geo}(G) = 2$ and $S = \{u, v\}$ be a minimum geodetic hub set of G. For $n = 2$, $G = K_2$ and $d(G) = 1$. For $n = 3$, $G = P_3$ and $d(G) = 2$. Let $n \geq 4$. For all v_i, $v_j \in V(G) \setminus S$ either v_i is adjacent to v_j or there exists $S-$path between v_i and v_j and every vertex of G lies on a $u - v$ geodesic path. Let $x, y \in V(G)$ with $d(x, y) = d(G) \geq 4$. We have the following cases.
Case 1: Let x, $y \in S$. Then $x = u$, $y = v$, u is not adjacent to v. Consider the shortest path $uw_1w_2 \cdots w_kv$, $k \geq 3$. Then w_1 and w_k do not have $S-$path, a contradiction to the fact that S is a hub set.
Case 2: Let $x, y \in V(G) \setminus S$. Since S is geodetic set, u is not adjacent to v. So, either x is adjacent to y or xuy or xvy is an $S-$path between x and y, so that $d(x, y) \geq 2$, a contradiction.
Case 3: Let $x \in S$, $y \in V(G) \setminus S$, let $x = u$. Since S is a geodetic set, x is not adjacent to v and y lies in a geodesic path $xw_1w_2 \cdots w_{1-l}yw_l \cdots w_kv$, say. Then, there is no $S-$path between w_1 and w_k, a contradiction to the fact that S is a hub set. Hence $d(x, y) \leq 3$, so $d(G) \leq 3$.

Lemma 2.7. *For $G = K_1 + \bigcup r_t K_t$ of order n, where r_t is any natural number and $\sum r_t \geq 2$, $h_{geo}(G) = n - 1$.*

Proof. Let $G = K_1 + \bigcup r_t K_t$, where $\sum r_t \geq 2$. Then $n \geq 3$ and G has exactly one cut vertex v, say, the remaining all vertices are extreme vertices. Then by Theorem 2.4, $S = V(G) \setminus \{v\}$ is a subset of any geodetic hub set of G and so $h_{geo}(G) = n - 1$.

Theorem 2.8. *Let $G = (n, m)$ be a connected graph. Then $h_{geo}(G) = n - 1$ if and only if G has a cut vertex w of degree $n - 1$ such that $G - w$ is the union of complete graphs.*

Proof. Let $h_{geo}(G) = n - 1$ and S be the minimum geodetic hub set such that $V(G) \setminus S = \{w\}$ for some $w \in V(G)$.

Step 1: If $S = \{v_1\}$ for $v_1 \in V(G)$ then $V(G) = \{w, v_1\}$. Since G is connected, $G = K_2$ for which $h_{geo}(G) = 2$, a contradiction. So, $h_{geo}(G) = |S| \geq 2$.

Step 2: If w lies on a cycle $wv_1v_2 \cdots v_kw$,say, then $k \neq 2$, for if $k = 2$ then w is not in any $v_1 - v_2$ geodesic path. Then $S \cup \{w\}$ is a minimum geodetic hub set of G, a contradiction. So, $k \geq 3$. Now, $S \setminus \{v_2\}$ would be a geodetic hub set, so that $h_{geo}(G) \leq n - 2$, a contradiction. Hence w is a cut vertex of G.

Step 3: If $deg(w) \leq n - 2$, then there exists $v_i \in S$ such that w is not adjacent to v_i, so, $S - \{v_i\}$ is a geodetic hub set, and $h_{geo}(G) \leq n - 2$, a contradiction. Hence $deg(w) = n - 1$.

Let H be a component of $G - w$ containing nonadjacent neighbors u and v of w, and $u = v_1v_2 \cdots v_k = v$ be a shortest $u - v$ path in H. Then $k \geq 3$ and $V(G) \setminus \{w, v_2\}$ is a geodetic hub set, a contradiction. Thus $N(w) \cap V(H)$ induces a complete graph. If $G - w$ consists of only one component then w is a extreme vertex, again a contradiction. Hence $G - w$ is disjoint union of complete graphs $H_1, H_2, \cdots, H_p$. The converse follows by Lemma 2.7.

Theorem 2.9. *If G is a connected noncomplete graph with $h(G) = 1$, then either $h_{geo}(G) = g(G)$ or $h_{geo}(G) = g(G) + 1$.*

Proof. Since $h(G) = 1$, let $\{x\}$ is a minimum hub set of G, $\Delta(G) \geq n - 2$ and $d(G) = 2$. G has at least two nonadjacent vertices. Let S be a minimum geodetic set of G. Then every $w \notin S$ belongs to a $u - v$ geodesic path for some $u, v \in S$. Since $d(G) = 2$, this geodesic path is uwv. Further, any nonadjacent vertices $x, y \in V(G) \setminus S$ are joined by a path xv_0y for some $v_0 \in S$, because $d(G) = 2$. Thus S is a minimum geodetic hub set of G and so $h_{geo}(G) = g(G)$. Now, if $x \in S$, then S itself is a geodetic hub set of G, so, $h_{geo}(G) = |S| = g(G)$. If $x \notin S$, then $S \cup \{x\}$ is a minimum geodetic hub set of G, so, $h_{geo}(G) = |S| + 1 = g(G) + 1$.

Remark 2.10. *The converse of Theorem 2.9 need not be true. For example, con-*

sider C_5, $h_{geo}(G) = g(G) = 3$ but $h(G) = 2$.

Theorem 2.11. *Let $G = K_{k_1,k_2,\ldots,k_n}$ be complete $n-$partite graph with $k_1 \leq k_2 \leq \ldots \leq k_n$. Then*

1. *if $k_i = 1$, for every i, $1 \leq i \leq n$, then $h_{geo}(G) = n$.*
2. *if $k_1 = 2$ or $k_1 = 3$, then $h_{geo}(G) = k_1$.*
3. *if $k_1 \geq 4$, then $h_{geo}(G) = 4$.*
4. *$k_i = 1$, $1 \leq i \leq t$ for some $t \geq 1$ and $2 \leq k_{t+1} \leq 4$, then $h_{geo}(G) = k_{t+1}$.*

Proof. Let $V(G) = V_1 \cup V_2 \cup \ldots \cup V_n$.

1. If $k_i = 1$, for every i, then $G = K_n$ and result follows from Theorem 2.5.
2. Every partite set V_i, $1 \leq i \leq n$ is a geodetic hub set. So, if $k_1 = 2$ or 3, then V_1 is the minimum geodetic hub set of G and so $h_{geo}(G) = k_1$.
3. Let $S^{''} = V_i$ or V_j, $1 \leq i, j \leq n$. $S^{''}$ is a geodetic hub set of G. But $S^{''}$ is not a minimum geodetic hub set of G. We show that $S = \{u, v, x, y\} \subset V_i \cup V_j$, where $u, v \in V_i$ and $x, y \in V_j$, for $i \neq j$, is a minimum geodetic hub set of G. Let $u, v \in S$ and $a, b \in V(G) \setminus S$. We consider the following cases.
   Case (i): If $a, b \in V_i$, $1 \leq i \leq n$, then axb is an $S-$path, a and b lie on $x - y$ geodesic path in G.
   Case (ii): If $a \in V_i$, $b \in V_j$, $1 \leq i, j \leq n$, then a and b are adjacent vertices. Also a lies on $x - y$ geodesic path and b lies on $u - v$ geodesic path in G.
   Case (iii): If $a, b \in V_j$, $1 \leq j \leq n$, then aub is an $S-$path in G and a and b lie on $u - v$ geodesic path in G.
   Case (iv): If $a, b \in V_m$, $1 \leq m \leq n$ and $m \neq i, j$, then aub is an $S-$path. Also a and b lie on $x - y$ geodesic path in G.
   In all the above cases S is a geodetic hub set of G. Since any three element set is not a geodetic hub set, we have $h_{geo}(G) = 4$.
4. The proof immediately follows from (2) and (3).

Theorem 2.12. *Let T be a tree of order $n \geq 3$. Then the following statements are equivalent.*

1. *$g(T) = h_{geo}(T)$.*
2. *T is either a star or a double star.*

3. $h(T) \leq 2$.

4. The set of all leaf vertices of T *is a hub set of* T.

Proof. Let S consists of all leaf vertices of T. From Theorem 1.2, S is the unique geodetic set of T.
(1)⇒ (2): Let $h_{geo}(T) = g(T)$. If T is neither a star nor a double star, then $d(T) > 3$. So, T has at least one nonleaf nonsupport vertex. Then there exists atleast two support vertices having no $S-$path and so $h_{geo}(G) > |S| = g(T)$, a contradiction. Hence T must be either a star or a double star.
(2)⇒ (3): If T is a star with central vertex v_0. Then $S = \{v_0\}$ is a minimum hub set of T. Hence $h(T) = 1$. If T be a double star with central vertices u_0 and v_0. Then $S = \{u_0, v_0\}$ is a minimum hub set of T. Hence $h(T) = 2$.
(3)⇒ (4): If $h(T) = 1$. Then there exists a vertex v_0, say, in T such that v_0 is a nonleaf vertex and all vertices other than v_0 are the leaf vertices in T. So S forms a hub set of T. If $h(T) = 2$. Then there exist nonleaf vertices u_0 and v_0 in T. Hence S forms a hub set of T.
(4) ⇒ (1): By hypothesis, S is a hub set of T, $h(T) \leq |S|$. Since S is the unique geodetic set of T, $g(T) = |S|$. Hence, S is a minimum geodetic hub set of T and so $g(T) = h_{geo}(T)$.

Proposition 2.13. *Let* $G_1, G_2, \ldots, G_t$ *be the components of a graph* G *with* $|V(G_i)| = n_i$ *and* $n_1 \leq n_2 \leq \ldots \leq n_t$. *Then* $h_{geo}(G) = h_{geo}(G_t) + \sum_{j=1}^{t-1} |V(G_j)|$.

Proof. Any minimum geodetic hub set S of G must contain all vertices of first $t-1$ components and the vertices of the geodetic hub set of the remaining component G_t. This means that minimum geodetic hub set of G is of the form $S = [\bigcup_{j=1}^{t-1} V(G_j)] \cup S_t$, where S_t is geodetic hub set of G_t. Thus, the conclusion follows.

Definition 2.14. *A geodetic hub set* S *in* G *is called a minimal geodetic hub set if no proper subset of* S *is a geodetic hub set of* G.

Remark 2.15. *Every minimum geodetic hub set of* G *is a minimal geodetic hub set of* G. *In Figure 1,* $S_1 = \{v_1, v_4\}$ *is a minimum geodetic hub set of* G. *But the converse need not be true because* $S_2 = \{v_2, v_3, v_5\}$ *is a minimal geodetic hub set of* G *and it is not a minimum geodetic hub set of* G.

Theorem 2.16. *Let* $G = G_1 + K_n$, *where* G_1 *is noncomplete connected graph and* $n \geq 1$. *Then* $h_{geo}(G) = min\{|S| : S \subset V(G_1) \text{ and } S \in \Omega(G)\}$, *where* $\Omega(G)$ *denotes the collection of all minimal geodetic hub sets of* G.

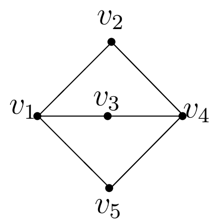

Figure 1

Proof. By the definition of join of graphs, $d(G) = 2$. Let $S \subset V(G)$ be a minimal geodetic hub set of G. Since G is noncomplete, there exists a pair of vertices $u, v \in S$ such that $d_G(u, v) = 2$. In particular, $u, v \in V(G_1)$. Also, each vertex of K_n lies on some $u - v$ geodesic path in G. We have $S \cap V(K_n) = \emptyset$. Hence $S \subset V(G_1)$. Thus the desired conclusion follows immediately.

Remark 2.17. *In Theorem 2.16, $\Omega(G)$ cannot be replaced by $\Omega(G_1)$. For example, any minimal geodetic hub set of $S_{n,m} + K_1$ cannot be a minimal geodetic hub set of $S_{n,m}$.*

Corollary 2.18. *Let $G = G_1 + K_n$, where G_1 is a noncomplete graph and $n \geq 1$. If $d(G_1) = 2$, then $h_{geo}(G) = h_{geo}(G_1)$.*
Proof. Since $d(G_1) = 2$, so, result follows by the Theorem 2.16.

Theorem 2.19. *Let $G_1 = (V, E)$ be a graph of order n and let G_2 be a graph. Let $G_2^1 = (V_1, E_1)$, $G_2^2 = (V_2, E_2), \ldots, G_2^n = (V_n, E_n)$ be n copies of G_2 in $G_1 \circ G_2$, such that $v_i \in V$ is adjacent to all vertices of G_2^i.*

1. *Given three different vertices x, y and v of $G_1 \circ G_2$, if $v \in V_i$ and $x, y \notin V_i$ then v does not lie on any $x - y$ geodesic path in $G_1 \circ G_2$.*

2. *If S is a geodetic hub set of $G_1 \circ G_2$, then $S \cap V_i \neq \emptyset$, for $1 \leq i \leq n$.*

3. *If S is a minimum geodetic hub set of $G_1 \circ G_2$, G_2 is noncomplete graph and $n = 1$ then $S \cap V = \emptyset$.*

4. *If G_2 is a noncomplete graph and S is a minimum geodetic hub set of $G_1 \circ G_2$, then for every i, $1 \leq i \leq n$, $S_i = S \cap V_i$ is a minimum geodetic hub set of $G_2^i + v_i$.*

5. *If G_2 is a noncomplete graph and for every i, $1 \leq i \leq n$, S_i is a minimum geodetic hub set of $G_2^i + v_i$, then $S_i \subseteq V_i$ and $S = \bigcup_{i=1}^{n} S_i \cup V$ is a minimum geodetic hub set of $G_1 \circ G_2$.*

Proof. The proof of **(1)** and **(2)** follow directly from the definition of geodetic hub set as the vertices belonging to V_i are adjacent to only one vertex not in V_i.
(3) Let S be a minimum geodetic hub set of $G_1 \circ G_2$. By (2), we have $S \cap V_i \neq \emptyset$ for every i, $1 \leq i \leq n$ and by (1), if $v \in V_i$, then there exist nonadjacent vertices $x, y \in S \cap V_i$ such that v lies on $x - y$ geodesic path. Since $d(G_2^i + v_i) \leq 2$, if $u, v \in V_i$, then uxv is an $S-$path in $G_2^i + v_i$. Suppose $S \cap V \neq \emptyset$, let $v_i \in S \cap V$, then $G_1 \circ G_2$ is a noncomplete graph. So there exist nonadjacent vertices $a, b \in S$ such that the vertex v_1 of G_1 belongs to $a - b$ geodesic path in $G_1 \circ G_2$ and v_1 is adjacent to each vertex in V_1. So, $S \setminus \{v_1\}$ is a geodetic hub set of $G_1 \circ G_2$, a contradiction. Hence $S \cap V = \emptyset$.
(4) Let G_2 be a noncomplete graph and S be a minimum geodetic hub set of $G_1 \circ G_2$. By (2) we have $S_i = S \cap V_i \neq \emptyset$ for every i, $1 \leq i \leq n$. Let $a \in (V_i \cup \{v_i\}) \setminus S_i$. We have the following cases.
Case (i): Let $a = v_i$. Since v_i is adjacent to every vertex of G_2^i and G_2^i is a noncomplete graph, there exist nonadjacent vertices c, d of G_2^i such that $c, d \in S_i$. Hence v_i lies on $c - d$ geodesic path in G_2^i.
Case (ii): Let $a \neq v_i$, then $a \in V_i$, and there exist nonadjacent vertices $x, y \in S_i$ such that a lies on $x - y$ geodesic path in G_2^i.
Also, v_i is adjacent to every vertex in G_2^i, by the definition of $G_1 \circ G_2$ and since S is a minimum geodetic hub set of $G_1 \circ G_2$, any two vertices of $(V_i \cup \{v_i\}) \setminus S_i$ have a S_i-path in $G_2^i + v_i$. In both the cases, S_i is a geodetic hub set of $G_2^i + v_i$. Suppose S_i is not a minimum geodetic hub set of $G_2^i + v_i$, then S is not a minimum geodetic hub set of $G_1 \circ G_2$, a contradiction. Hence, (4) follows.
(5) Let G_2 be a noncomplete graph, for every i, $1 \leq i \leq n$, S_i be a minimum geodetic hub set of $G_2^i + v_i$. Since $G_2^i + v_i$ is noncomplete and v_i is adjacent to every vertex of G_2^i, there exist nonadjacent vertices x, y of G_2^i such that $x, y \in S_i$. Hence v_i lies on $x - y$ geodesic path. Therefore $v_i \notin S_i$ and so, $S_i \subseteq V_i$. Now, let $S = \bigcup_{i=1}^{n} S_i \cup V$ and $a, b \in V(G_1 \circ G_2) \setminus S$. We consider the following cases.
Case (i): Suppose $a, b \in V_i$. Since S_i is a geodetic hub set of $G_2^i + v_i$, there exists vertices $x, y \in S_i$ such that a and b lie on $x - y$ geodesic path in $G_1 \circ G_2$ and axb or ayb is an $S-$path in $G_1 \circ G_2$.
Case (ii): Suppose $a \in V_i$ and $b \in V_j$, for $i \neq j$. By case (i) $axv_1v_2 \cdots v_tyb$ for some t, $1 \leq t \leq n$, is an $S-$path, for some $x \in S_i$ and $y \in S_j$. Also, a and b lie on some geodesic path between vertices in $G_1 \circ G_2$.
Hence, in both the cases, S is a geodetic hub set of $G_1 \circ G_2$. Since S_i is a minimum geodetic hub set of $G_2^i + v_i$, for every $1 \leq i \leq n$, S is a minimum geodetic hub set of $G_1 \circ G_2$.

Corollary 2.20. *Let G_1 be a graph of order n and G_2 be a noncomplete graph. Then S is a minimum geodetic hub set of $G_1 \circ G_2$ if and only if $S = \bigcup_{i=1}^{n} S_i \cup V$, where each $S_i \subseteq V_i$ is a minimum geodetic hub set of $G_2^i + v_i$.*

Proof. Result follows from (4) and (5) of Theorem 2.19.

Theorem 2.21. *For any two graphs G_1 and G_2, with $|V(G_1)| = n_1$ and $|V(G_2)| = n_2$*

$$h_{geo}(G_1 \circ G_2) = \begin{cases} n_1 + n_1 n_2, & \text{if } G_1 \text{ and } G_2 \text{ both complete;} \\ n_1 n_2 + h(G_1), & \text{if } G_1 \text{ is noncomplete and } G_2 \text{ is complete.} \end{cases}$$

Proof. Suppose G_1 and G_2 are complete then $G_1 \circ G_2$ is also complete then by Theorem 2.5 $h_{geo}(G_1 \circ G_2) = n_1 + n_1 n_2$.

Suppose G_1 is any noncomplete graph and G_2 is complete. Let S be a minimum geodetic hub set of G, by the definition of corona product and geodetic hub set, S has all vertices of n_1 copies of $V(G_2)$ together with all vertices of minimum hub set of G_1.

So, $h_{geo}(G_1 \circ G_2) = n_1 n_2 + h(G_1)$.

Theorem 2.22. *If a and b are positive integers such that $2 \leq a \leq b$, then there exists a connected graph G of order b with $h_{geo}(G) = a$.*

Proof. If $a = b$ and $G = K_b$, then by Theorem 2.5 $h_{geo}(G) = a$. If $a < b$, let $H = K_a$ be the complete graph on a vertices $v_1, v_2, \ldots, v_a$. Add $b - a + 1$ new vertices $u_1, u_2, \ldots, u_{b-a}, w$ to H and join the vertices $u_1, u_2, \ldots, u_{b-a}$ to both v_a and w. We get the graph G shown in Figure 2.

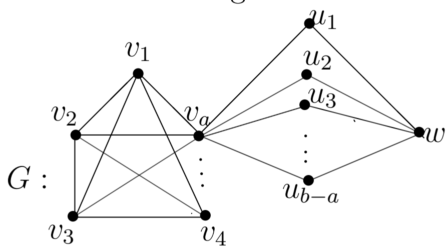

Figure 2

Now $S = \{v_1, v_2, \ldots, v_{a-1}\}$ consists of all extreme vertices of G, by Theorem 1.1, they must belong to every geodetic set, $S_1 = S \cup \{w\}$ is a geodetic set and also a minimum geodetic hub set of G. $h_{geo}(G) = |S_1| = a$.

Theorem 2.23. *For any two positive integers a and b with $2 \leq a \leq b$, there exists a connected graph G with $g(G) = a$ and $h_{geo}(G) = b$.*

Proof. Consider a path P_{b-a+3}, together with $(a-1)$ end vertices all adjacent to the same end vertex v_{b-a+3} of P_{b-a+3}, resultant graph G is shown in Figure 3. Clearly $S = \{v_1, u_1, u_2, \ldots, u_{a-1}\}$ is the set of all extreme vertices of G. Since G is a tree, by Theorem 1.1 $g(G) = a$. The set $S^{'} = \{v_1, v_2, \ldots, v_{b-a+1}, u_1, u_2, \ldots, u_{a-1}\}$ is a minimum geodetic hub set of G with $|S^{'}| = b$.

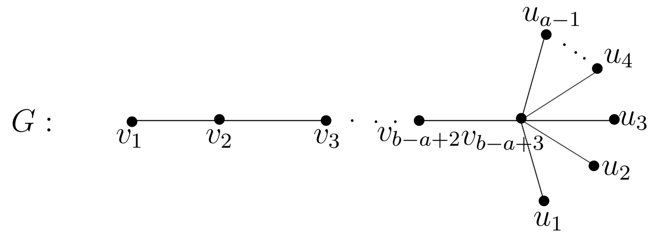

Figure 3

Theorem 2.24. *Let $a \geq 4$ and $b \geq 2$ be any two integers. Then there is a connected graph G with $h(G) = a$, $g(G) = b$ and $h_{geo}(G) = a + b$.*
Proof. Consider a cycle C_6 with vertex set $V(C_6) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and a path P_{a-3} with vertex set $V(P_{a-3}) = \{u_1, u_2, \ldots, u_{a-3}\}$. Obtain graph H from C_6 and P_{a-3} by joining the vertex v_4 in C_6 and u_1 in P_{a-3}. Add $b-1$ new vertices $w_1, w_2, \ldots, w_{b-1}$ to H and join each of them to the vertex v_1. The resultant graph G is shown in Figure 4.

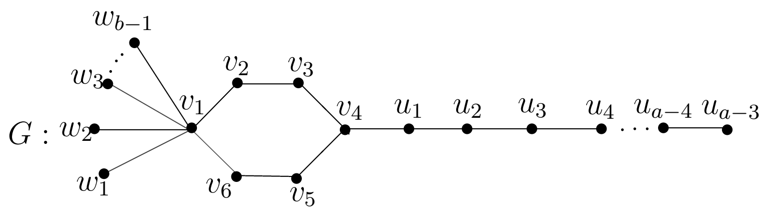

Figure 4

The set $S = \{v_1, v_2, v_3, v_4, u_1, u_2, \ldots, u_{a-4}\}$ is a minimum hub set of G with $|S| = a$. Let $S^{'} = \{w_1, w_2, \ldots, w_{b-1}, u_{a-3}\}$ be the set of all extreme vertices of G so that they must belong to every geodetic set of G. $S^{'}$ itself is a geodetic set of G and hence $S^{'}$ is a minimum geodetic set of G with $|S^{'}| = b$. Thus $h(G) = a$ and $g(G) = b$. Clearly $S \cup S^{'}$ is a minimum geodetic hub set and so $h_{geo}(G) = a + b$.
Theorem 2.25. *Every pair of integers a, b with $1 \leq a < b$ can be realized as the hub number and the geodetic hub number, respectively, of some connected graph G.*
Proof. We prove this theorem by considering two cases.
Case (i): $1 = a < b$. Let $G = K_{1,b}$, $b \geq 2$ be a star. Then $h(G) = 1$ and

$h_{geo}(G) = b$.

Case (ii): $2 \leq a < b$. Take $H = C_{a+2}$, the cycle graph with $a + 2$ vertices $v_1, v_2, \ldots, v_{a+2}$. Add $b - a + 1$ new vertices $u_1, u_2, \ldots u_{b-a+1}$ to H by joining each of them to v_1 to get graph G of Figure 5.

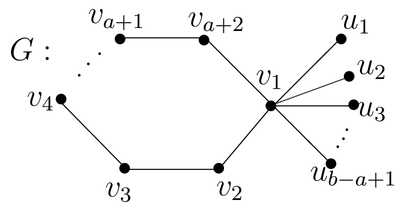

Figure 5

Clearly the set $S = \{v_1, v_2, \ldots, v_a\}$ is minimum hub set of G with $|S| = a$. Let $S^{'} = \{v_2, v_3, \ldots, v_a, u_1, u_2, \ldots, u_{b-a+1}\}$ be the set of all extreme vertices of G so that by Theorem 1.1, they must belong to every geodetic set of G. Also $S^{'}$ itself is a geodetic set of G and hence $S^{'}$ is a minimum hub set of G. So, $S^{'}$ is a minimum geodetic hub set of G with $|S^{'}| = b$.

3. Conclusion

In this paper, the concept of geodetic hub number is introduced and studied it for several classes of graphs and graph operations.

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